Table 1 Frequencies vs mode numbers

Number	Ω^a (this analysis)	Ω^a (Ref. 4)	n^b	m^b
1	3.1962	3.1960	0	0
2	4.6109	4.6110	1	0
3	5.9057	5.9057	2	0
4	6.3064	6.3064	0	1
5	7.1435	7.1435	3	0
6	7.7993	7.7993	1	1
7	8.3466	8.3466	4	0
8	9.1969	9.1969	2	1
9	9.4395	9.4395	0	2
10	9.5257	9.5257	5	0
11	10.5367	10.5367	3	1
12	10.6870	10.6870	6	0
13	10.9581	10.9581	1	2
14	11.8345	11.8345	7	0
15	11.8367	11.8367	4	1
16	12.4022	12.4022	2	2
17	12.5771	12.5771	0	3
18	12.9709	12.9770	8	0
19	13.1074	13.1074	5	1
20	13.7951	13.7951	3	2
21	14.0981	14.1086	9	0
22	14.1086	15.5795	1	3
23	14.3552	15.7164	6	1
24	15.1499	17.2557	4	2
25	15.2175	18.7440	10	0
26	15.5795	18.8565	2	3
27	15.5846	20.4010	7	1
28	15.7164	21.9977	0	4
29	16.3303	23.5453	11	0
30	16.4751	25.1379	5	2

[a] Ω defined in Eq. (5) of Ref. 1. [b] n and m correspond to frequencies determined in this analysis.

where P is the applied concentrated load.

Note that a comparison of Eq. (6) with Eq. (25) of Ref. 1 demonstrates that the factor of $(1/r)$ has been omitted from the right-hand side of the latter equation. Results obtained in the present analysis for the above half-sine pulses are plotted in Fig. 1 using the dimensionless variables introduced in Ref. 4. The variables associated with the ordinate are $W=\bar{W}/\bar{a}$ and $P=\bar{P}/(\bar{a}C)$ with the $\bar{W}$ the transverse displacement, $\bar{P}$ [not P as in Eq. (6)] the applied concentrated load, $\bar{a}$ the plate radius, and c the extensional rigidity. P is defined incorrectly in the Appendix of Ref. 4, but was subsequently defined as given above the Kunukkasseril and Chandrasekharan.[8] The abscissa is obtained by dividing the elapsed time by the period of the fundamental mode of the plate ($T_f=5.03\times10^{-3}$ s). Due to the normalization by the applied concentrated load, the static solution from Ref. 5, shown as a solid horizontal line in Fig. 1, is identical for these three loadings. The discrepancies between Figs. 1 and Fig. 3 of Ref. 4 are attributed to:

1) The use of the incorrect normalization condition

$$\int_0^{2\pi}\int_0^{1} W_{nm}^2 r\mathrm{d}r\mathrm{d}\theta=\pi$$

in Eq. (8) of Ref. 4 instead of that given by Eq. (3).

2) The omission of several modes in the tabulation of the first 30 modes given in Table 1 of Ref. 4. For comparison, the lowest 30 modes employed in the present analysis as well as those given in Ref. 4 are tabulated in Table 1.

The effect of increasing the number of modes in the solution from 30 to 60 is barely perceptible (the curves of Fig. 1, where the 60-mode solutions are plotted, are changed by less than 1%). As a further check on the solution, the 22.6-kg load was applied in the quasistatic manner by specifying the dimensionless duration of impact as $(1\times10^7)t/T_f$. As required, the maximum response occurred at $t/T_f=0.5\times10^7$ and was 0.99 of that given by Ref. 5.

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Effect of Nonhomogeneity in Orthotropic Curved Plates

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Introduction

In the design of fiber-reinforced plastic structures, it is often necessary to use fibers with different modulus and strength properties (mixing and matching method) in order to minimize the extreme fiber stress. Curved members of uniform thickness and such type of mixed fiber structure may then be approximated by radially nonhomogenous orthotropic curved plates. The problem of an orthotropic homogenous plate was first attempted by Carrier.[1] Later, Bert[2] and Shaffer[3,4] obtained the general solutions for nonhomogenous disks and tubes. Using linear elasticity theory, Griffin[5] studied the bending behavior of an isotropic homogenous thick curved plate due to edge loadings, and compared his results with elementary theory.

In this paper, the bending behavior of a nonhomogenous orthotropic curved plate subjected to surface tractions and edge forces is investigated. The layered fiber reinforcement in the plate is assumed to be radially nonhomogenous and orthotropic, and the Airy's stress function approach is used.

Analysis

The stress-strain relations for an orthotropic and radially nonhomogenous elastic material can be written in the form[6]:

$$\epsilon_r=a_{11}\sigma_r+a_{12}\sigma_\theta,\quad \epsilon_\theta=a_{12}\sigma_r+a_{22}\sigma_\theta,\quad \gamma_{r\theta}=a_{66}\tau_{r\theta} \tag{1}$$

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where

$$a_{11}=\frac{1}{E_r}\left(1-\frac{E_z}{E_r}\nu_{rz}^2\right),\quad a_{22}=\frac{1}{E_\theta}\left(1-\frac{E_z}{E_\theta}\nu_{\theta z}^2\right),$$

$$a_{12}=-\frac{\nu_{r\theta}}{E_r}-\frac{\nu_{rz}}{E_r}\frac{\nu_{\theta z}}{E_\theta}E_z,\quad a_{66}=\frac{1}{G_{r\theta}} \tag{2}$$

for plane strain assumption. The moduli of elasticity and Poisson's ratios are related by the expressions

$$E_r\nu_{r\theta}=E_\theta\nu_{\theta r},\quad E_r\nu_{rz}=E_z\nu_{zr},\quad E_\theta\nu_{\theta z}=E_z\nu_{z\theta} \tag{3}$$

and the material is radially nonhomogenous in the sense that

$$E_r(r)=E_1r^n,\quad E_\theta(r)=E_2r^n,\quad E_z(r)=E_3r^n \tag{4}$$

It can be seen from Eqs. (3) and (4) that the Poisson's ratios are only related to the material constants E_1, E_2, and E_3.

In the absence of body forces, the equations of equilibrium are identically satisfied if the stresses are derived from the Airy stress function $\phi(r,\theta)$, which is defined by

$$\sigma_r=\frac{1}{r}\frac{\partial\phi}{\partial r}+\frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}$$

$$\sigma_\theta=\frac{\partial^2\phi}{\partial r^2},\quad \tau_{r\theta}=-\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) \tag{5}$$

Substituting Eq. (5) into Eq. (1), and the resulting expressions for the strains into the compatibility equation

$$\left(\frac{1}{r^2}\frac{\partial^2\epsilon_r}{\partial\theta^2}-\frac{1}{r}\frac{\partial\epsilon_r}{\partial r}\right)+\frac{1}{r}\frac{\partial^2(r\epsilon_\theta)}{\partial r^2}=\frac{1}{r^2}\frac{\partial^2(r\gamma_{r\theta})}{\partial r\partial\theta} \tag{6}$$

one obtains the governing equation in terms of the stress function $\phi(r,\theta)$ as

$$b_{22}\frac{\partial^4\phi}{\partial r^4}+\frac{2(1-n)}{r}\frac{\partial^3\phi}{\partial r^3}-\frac{(b_{11}+nb_{12}+n(1-n)b_{22})}{r^2}\frac{\partial^2\phi}{\partial r^2}$$
$$+\frac{(n+1)b_{11}+n(n+1)b_{12}}{r^3}\frac{\partial\phi}{\partial r}$$
$$+\frac{(n+2)b_{11}+(n+1)(n+2)b_{12}+(n+1)b_{66}}{r^4}\frac{\partial^2\phi}{\partial\theta^2}$$
$$-\frac{2(n+1)b_{12}+(n+1)b_{66}}{r^3}\frac{\partial^3\phi}{\partial r\partial\theta^2}+\frac{2b_{12}b_{66}}{r^2}\frac{\partial^4\phi}{\partial r^2\partial\theta^2}$$
$$+\frac{b_{11}}{r^4}\frac{\partial^4\phi}{\partial\theta^4}=0 \tag{7}$$

where

$$b_{11}=a_{11}r^n=\frac{1}{E_1}\left[1-\left(\frac{E_3}{E_1}\right)\nu_{rz}^2\right]$$

$$b_{12}=a_{12}r^n=\frac{1}{E_1}\left[-\nu r_\theta-\left(\frac{E_3}{E_2}\right)\nu_{rz}\nu_{\theta z}\right]$$

$$b_{22}=a_{22}r^n=\frac{1}{E_2}\left[1-\left(\frac{E_3}{E_2}\right)\nu_{\theta z}^2\right]$$

$$b_{66}=a_{66}r^n=\frac{1}{E_1}\left[1+\left(\frac{E_1}{E_2}\right)+2\nu_{r\theta}\right] \tag{8}$$

A. Symmetrical Bending Problems

When the curved plate is subjected to a pure bending moment M acting on the two edges, the problem becomes rotationally symmetric and Eq. (7) can be reduced to a linear differential equation with constant coefficients

$$\frac{d^4\phi}{dt^4}-2(n+2)\frac{d^3\phi}{dt^3}+\left[n^2+n\left(5-\frac{b_{12}}{b_{22}}\right)+\left(5-\frac{b_{11}}{b_{22}}\right)\right]\frac{d^2\phi}{dt^2}+\left[n^2\left(\frac{b_{12}}{b_{22}}-1\right)+n\left(\frac{b_{11}}{b_{22}}+2\frac{b_{12}}{b_{22}}-3\right)+2\left(\frac{b_{11}}{b_{22}}-1\right)\right]\frac{d\phi}{dt}=0 \tag{9}$$

where the new independent variable t is related to r by the transformation equation $t=\ell n r$. The general solution of Eq. (9) takes the form

$$\phi(r)=A+Br^\alpha+Cr^\beta+Dr^\gamma \tag{10}$$

where the exponents α, β, and γ are roots of the characteristic equation of Eq. (9). Constants A, B, C, and D are determined from the boundary conditions

$$\sigma_r=0 \text{ at } r=r_1 \text{ and } r=r_2,\quad \int_{r_1}^{r_2}\sigma_\theta r\,dr=-M \tag{11}$$

and the stress components are obtained as

$$\sigma_r=\left(\frac{\alpha\beta\gamma M}{r_1{}^2R}\right)[(k^\beta-k^{\alpha+\beta})b^{\alpha-2}-(k^\alpha-k^{\alpha+\beta})b^{\beta-2}-(k^\beta-k^\alpha)b^{\gamma-2}]$$

$$\sigma_\theta=\left(\frac{\alpha\beta\gamma M}{r_1{}^2R}\right)[(\alpha-1)(k^\beta-k^{\alpha+\beta})b^{\alpha-2}-(\beta-1)(k^\alpha-k^{\alpha+\beta})b^{\beta-2}-(\gamma-1)(k^\beta-k^\alpha)b^{\gamma-2}] \tag{12}$$

where

$k=r_2/r_1,\qquad b=r/r_1$

$R=\alpha\gamma(\beta-1)(2k^{\alpha+\beta}-k^\alpha-k^{\alpha+2\beta})-\beta\gamma(\alpha-1)\times(2k^{\alpha+\beta}-k^\beta-k^{2\alpha+\beta})+\alpha\beta(\gamma-1)(k^{\beta+\gamma}-k^{\alpha+\gamma}-k^\beta+k^\alpha)$

Since zero and $\gamma=n+2$ are two roots of the characteristic equation of Eq. (9), the other two roots, α and β, can be determined easily.

B. Nonsymmetrical Bending Problems

When the end loads, such as normal and shear forces, are applied to the curved plate, or when the plate is subjected to nonuniform tractions at the inner and outer radial surfaces, Eq. (7) may be solved using the separate of variables method and the transformation of the independent variable $t=\ell n r$. For a curved plate loaded by a shear at the end of plate and a nonuniform contact pressure induced by bending contact at the inner surface, the contact pressure can be assumed in the form $p=p_0\cos\theta$. The stress function may then be expressed as

$$\Phi(r,\theta)=f(r)\cos\theta \tag{13}$$

From Eqs. (7) and (13), it can easily be shown that the function $f(r)$ satisfies

$$\frac{d^4f}{dt^4}-2(n+2)\frac{d^3f}{dt^3}+\left[n^2+n\left(5-\frac{b_{12}}{b_{22}}\right)\right.$$

$$\left.+\left(5-\frac{b_{11}}{b_{22}}-2\frac{b_{12}}{b_{22}}-\frac{b_{66}}{b_{22}}\right)\right]\frac{d^2f}{dt^2}+\left[n^2\left(\frac{b_{12}}{b_{22}}-1\right)\right.$$

$$+n\left(\frac{b_{12}}{b_{22}}+4\frac{b_{12}}{22}+\frac{b_{66}}{b_{22}}-3\right)+2\left(\frac{b_{11}}{b_{22}}+2\frac{b_{12}}{b_{22}}+\frac{b_{66}}{b_{22}}\right.$$

$$\left.\left.-1\right)\right]\frac{df}{dt}+\left[n^2\frac{b_{12}}{b_{22}}-n\left(\frac{b_{11}}{b_{22}}+3\frac{b_{12}}{b_{22}}+\frac{b_{66}}{b_{22}}\right)\right.$$

$$\left.-\left[\left(\frac{b_{11}}{b_{22}}+2\frac{b_{12}}{b_{22}}+\frac{b_{66}}{b_{22}}\right)\right]f(t)=0 \tag{14}$$

The solution of Eq. (14) gives

$$\phi(r,\theta)=(Er^{\delta}+Fr^{\mu}+Gr+Hr^{\rho})\cos\theta \tag{15}$$

where the characteristic roots δ and μ are related to the orthotropy of the material, and ρ is equal to $n+1$ for all values of n. Applying the stress boundary conditions,

$$\sigma_r=-p_0\cos\theta \qquad \text{at} \quad r=r_1$$

$$\sigma_r=0 \qquad \text{at} \quad r=r_2$$

$$\int_{r_1}^{r_2}\tau_{r\theta}dr=P \qquad \text{at} \quad \theta=\pi/2 \tag{16}$$

the stress field for the nonsymmetric loading case is obtained as

$$\sigma_r=\frac{p_0b^{\rho-2}}{N'(1-k^{\delta-\rho})}[(\delta-1)(\mu-1)(1-k^{\delta-\rho})\lambda M'$$

$$+(\mu-1)(k^{\delta-1}-1)M'-\frac{(\delta-1)(\mu-1)}{(\rho-1)}(k^{\delta-1}-k^{\delta-\rho})M'$$

$$-(b^{\delta-\rho}-k^{\delta-\rho})N']\cos\theta$$

$$\sigma_\theta=\frac{p_0b^{\rho-2}}{N'(1-k^{\delta-\rho})}[(\delta-1)(\mu-1)(1-k^{\delta-\rho})\lambda L'$$

$$+(\mu-1)(k^{\delta-1}-1)L'-\frac{(\delta-1)(\mu-1)}{(\rho-1)}(k^{\delta-1}-k^{\delta-\rho})L'$$

$$-(\delta b^{\delta-\rho}-\rho k^{\delta-\rho})N']\cos\theta$$

$$\tau_{r\theta}=\sigma_r\tan\theta \tag{17}$$

where

$$M'=(1-k^{\delta-\rho})(b^{\mu-\rho}-k^{\mu-\rho})-(1-k^{\mu-\rho})(b^{\delta-\rho}-k^{\delta-\rho})$$

$$L'=(1-k^{\delta-\rho})(\mu b^{\mu-\rho}-\rho k^{\mu-\rho})$$

$$-(1-k^{\mu-\rho})(\delta b^{\delta-\rho}-\rho k^{\delta-\rho})$$

$$N'=(\delta-1)(1-k^{\delta-\rho})(k^{\mu-1}-1)-(\mu-1)(1-k^{\mu-\rho})$$

$$\times(k^{\delta-1}-1)+\frac{(\delta-1)(\mu-1)}{(\rho-1)}(k^{\rho-1}-1)[k^{\delta-\rho}(1-k^{\mu-\rho})$$

$$-k^{\mu-\rho}(1-k^{\delta-\rho})]$$

and λ is a parameter relating the contact pressure at the inner surface to the applied shear force by the expression $P=\lambda r_1p_0$.

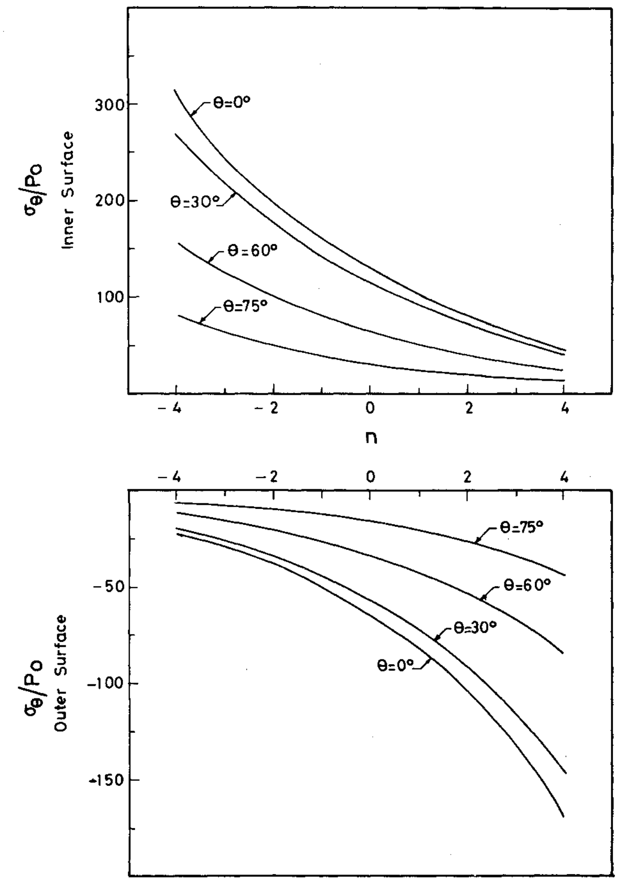

Fig. 1 Variation of hoop stresses at $r=r_2$ and $r=r_1$ for nonsymmetrical bending, $r_2=2r_1$.

The effect of material anisotropy and nonhomogeneity on the stress field was investigated for a set of elastic constants $E_1/E_2=4$, $E_1/E_3=2$, and $E_2/E_3=0.5$. For both symmetrical and nonsymmetrical cases, the maximum radial stress increases significantly for $-n$ and decreases for $+n$. The circumferential stress plot given in Fig. 1 for nonsymmetrical bending with $\lambda=10$ shows that σ_θ at the inner surface is lower (than $n=0$ case) when the parameter n is positive for all values of θ. On the other hand, the circumferential stress is higher at the outer surface for the same positive value of n. Thus, the results of the preceding analysis may be used to minimize the extreme fiber stress on one radial surface, while keeping the stress level on the other surface as low as possible. The detailed results of analysis, as well as numerical comparison of n values, are given in Ref. 7.

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